

# THE NUMBER OF UNIMODULAR ZEROS OF THE RUDIN-SHAPIRO POLYNOMIALS (PRELIMINARY VERSION)

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ABSTRACT. We show that the Rudin-Shapiro polynomials  $P_n$  and  $Q_n$  of degree  $N - 1$  with  $N := 2^n$  have  $o(N)$  zeros on the unit circle. This should be compared with a result of B. Conrey, A. Granville, B. Poonen, and K. Soundararajan stating that for odd primes  $p$  the Fekete polynomials  $f_p$  of degree  $p - 1$  have asymptotically  $\kappa_0 p$  zeros on the unit circle, where  $0.500813 > \kappa_0 > 0.500668$ . Our approach is based heavily on the Saffari and Montgomery conjectures proved recently by B. Rodgers.

## 1. INTRODUCTION AND NOTATION

Let  $D := \{z \in \mathbb{C} : |z| < 1\}$  denote the open unit disk of the complex plane. Let  $\partial D := \{z \in \mathbb{C} : |z| = 1\}$  denote the unit circle of the complex plane. The Mahler measure  $M_0(Q)$  is defined for bounded measurable functions  $Q$  on  $\partial D$  by

$$M_0(Q) := \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \log |Q(e^{it})| dt \right).$$

It is well known, see [HL-52], for instance, that

$$M_0(Q) = \lim_{q \rightarrow 0+} M_q(Q),$$

where

$$M_q(Q) := \left( \frac{1}{2\pi} \int_0^{2\pi} |Q(e^{it})|^q dt \right)^{1/q}, \quad q > 0.$$

It is a simple consequence of the Jensen formula that

$$M_0(Q) = |c| \prod_{k=1}^n \max\{1, |z_k|\}$$

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for every polynomial of the form

$$Q(z) = c \prod_{k=1}^n (z - z_k), \quad c, z_k \in \mathbb{C}.$$

See [BE-95, p. 271] or [B-02, p. 3], for instance. Let  $D := \{z \in \mathbb{C} : |z| < 1\}$  denote the open unit disk of the complex plane. Let  $\partial D := \{z \in \mathbb{C} : |z| = 1\}$  denote the unit circle of the complex plane. Let  $\mathcal{P}_n^c$  be the set of all algebraic polynomials of degree at most  $n$  with complex coefficients. Let  $\mathcal{T}_n$  be the set of all real (that is, real-valued on the real line) trigonometric polynomials of degree at most  $n$  with real coefficients.

Finding polynomials with suitably restricted coefficients and maximal Mahler measure has interested many authors. The classes

$$\mathcal{L}_n := \left\{ p : p(z) = \sum_{k=0}^n a_k z^k, \quad a_k \in \{-1, 1\} \right\}$$

of Littlewood polynomials and the classes

$$\mathcal{K}_n := \left\{ p : p(z) = \sum_{k=0}^n a_k z^k, \quad a_k \in \mathbb{C}, \quad |a_k| = 1 \right\}$$

of unimodular polynomials are two of the most important classes considered. Observe that  $\mathcal{L}_n \subset \mathcal{K}_n$  and

$$M_0(Q) \leq M_2(Q) = \sqrt{n+1}$$

for every  $Q \in \mathcal{K}_n$ . Beller and Newman [BN-73] constructed unimodular polynomials  $Q_n \in \mathcal{K}_n$  whose Mahler measure  $M_0(Q, [0, 2\pi])$  is at least  $\sqrt{n} - c/\log n$ .

Section 4 of [B-02] is devoted to the study of Rudin-Shapiro polynomials. Littlewood asked if there were polynomials  $p_{n_k} \in \mathcal{L}_{n_k}$  satisfying

$$c_1 \sqrt{n_k + 1} \leq |p_{n_k}(z)| \leq c_2 \sqrt{n_k + 1}, \quad z \in \partial D,$$

with some absolute constants  $c_1 > 0$  and  $c_2 > 0$ , see [B-02, p. 27] for a reference to this problem of Littlewood. To satisfy just the lower bound, by itself, seems very hard, and no such sequence  $(p_{n_k})$  of Littlewood polynomials  $p_{n_k} \in \mathcal{L}_{n_k}$  is known. A sequence of Littlewood polynomials that satisfies just the upper bound is given by the Rudin-Shapiro polynomials. The Rudin-Shapiro polynomials appear in Harold Shapiro's 1951 thesis [S-51] at MIT and are sometimes called just Shapiro polynomials. They also arise independently in Golay's paper [G-51]. They are remarkably simple to construct and are a rich source of counterexamples to possible conjectures.

The Rudin-Shapiro polynomials are defined recursively as follows:

$$P_0(z) := 1, \quad Q_0(z) := 1,$$

and

$$\begin{aligned} P_{n+1}(z) &:= P_n(z) + z^{2^n} Q_n(z), \\ Q_{n+1}(z) &:= P_n(z) - z^{2^n} Q_n(z), \end{aligned}$$

for  $n = 0, 1, 2, \dots$ . Note that both  $P_n$  and  $Q_n$  are polynomials of degree  $N - 1$  with  $N := 2^n$  having each of their coefficients in  $\{-1, 1\}$ . It is well known and easy to check by using the parallelogram law that

$$|P_{n+1}(z)|^2 + |Q_{n+1}(z)|^2 = 2(|P_n(z)|^2 + |Q_n(z)|^2), \quad z \in \partial D.$$

Hence

$$(1.1) \quad |P_n(z)|^2 + |Q_n(z)|^2 = 2^{n+1} = 2N, \quad z \in \partial D.$$

It is also well known (see Section 4 of [B-02], for instance), that

$$(1.2) \quad |Q_n(z)| = |P_n(-z)|, \quad z \in \partial D.$$

Peter Borwein's book [B-02] presents a few more basic results on the Rudin-Shapiro polynomials. Cyclotomic properties of the Rudin-Shapiro polynomials are discussed in [BL-76]. Obviously  $M_2(P_n, [0, 2\pi]) = 2^{n/2}$  by the Parseval formula. In 1968 Littlewood [L-68] evaluated  $M_4(P_n, [0, 2\pi])$  and found that  $M_4(P_n, [0, 2\pi]) \sim (4^{n+1}/3)^{1/4}$ . Rudin-Shapiro like polynomials in  $L_4$  on the unit circle of the complex plane are studied in [BM-00].

P. Borwein and Lockhart [BL-01] investigated the asymptotic behavior of the mean value of normalized  $L_p$  norms of Littlewood polynomials for arbitrary  $p > 0$ . They proved that

$$\lim_{n \rightarrow \infty} \frac{1}{2^{n+1}} \sum_{f \in \mathcal{L}_n} \frac{(M_q(f, [0, 2\pi]))^q}{n^{q/2}} = \Gamma\left(1 + \frac{q}{2}\right).$$

An analogue of this result for  $q = 0$  (the Mahler measure) has been proved recently in [CE-15c]. Similar results on the average Mahler measure and  $L_q(\partial D)$  norms of unimodular polynomials  $P \in \mathcal{K}_n$  had been established earlier in [CM-11].

In 1980 Saffari conjectured the following.

**Conjecture 1.1.** *Let  $P_n$  and  $Q_n$  be the Rudin-Shapiro polynomials of degree  $N - 1$  with  $N := 2^n$ . We have*

$$M_q(P_n, [0, 2\pi]) = M_q(Q_n, [0, 2\pi]) \sim \frac{2^{(n+1)/2}}{(q/2 + 1)^{1/q}}$$

for all real exponents  $q > 0$ . Equivalently, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} m\left(\left\{t \in K : \left|\frac{P_n(e^{it})}{\sqrt{2^{n+1}}}\right|^2 \in [\alpha, \beta]\right\}\right) \\ &= \lim_{n \rightarrow \infty} m\left(\left\{t \in K : \left|\frac{Q_n(e^{it})}{\sqrt{2^{n+1}}}\right|^2 \in [\alpha, \beta]\right\}\right) = \beta - \alpha \end{aligned}$$

whenever  $0 \leq \alpha < \beta \leq 1$ .

This conjecture was proved for all even values of  $q \leq 52$  by Doche [D-05] and Doche and Habsieger [DH-04]. Recently B. Rodgers [B-16] proved Saffari's Conjecture 1.1 for all  $q > 0$ . An extension of Saffari's conjecture is Montgomery's conjecture below.

**Conjecture 1.2.** *Let  $P_n$  and  $Q_n$  be the Rudin-Shapiro polynomials of degree  $N - 1$  with  $N := 2^n$ . We have*

$$\begin{aligned} & \lim_{n \rightarrow \infty} m \left( \left\{ t \in K : \frac{P_n(e^{it})}{\sqrt{2^{n+1}}} \in E \right\} \right) \\ &= \lim_{n \rightarrow \infty} m \left( \left\{ t \in K : \frac{Q_n(e^{it})}{\sqrt{2^{n+1}}} \in E \right\} \right) = \frac{1}{\pi} m(E) \end{aligned}$$

for any measurable set  $E \subset D := \{z \in \mathbb{C} : |z| < 1\}$ .

B. Rodgers [B-16] proved Montgomery's Conjecture 1.2 as well.

Despite the simplicity of their definition not much is known about the Rudin-Shapiro polynomials. It has been shown in [E-16c] fairly recently that the Mahler measure ( $L_0$  norm) and the  $L_\infty$  norm of the Rudin-Shapiro polynomials  $P_n$  and  $Q_n$  of degree  $N - 1$  with  $N := 2^n$  on the unit circle of the complex plane have the same size, that is, the Mahler measure of the Rudin-Shapiro polynomials of degree  $N - 1$  with  $N := 2^n$  is bounded from below by  $cN^{1/2}$ , where  $c > 0$  is an absolute constant.

It is shown in this paper that the Rudin-Shapiro polynomials  $P_n$  and  $Q_n$  of degree  $N - 1$  with  $N := 2^n$  have  $o(N)$  zeros on the unit circle.

For a prime number  $p$  the  $p$ -th Fekete polynomial is defined as

$$f_p(z) := \sum_{k=1}^{p-1} \left( \frac{k}{p} \right) z^k,$$

where

$$\left( \frac{k}{p} \right) = \begin{cases} 1, & \text{if } x^2 \equiv k \pmod{p} \text{ has a nonzero solution,} \\ 0, & \text{if } p \text{ divides } k, \\ -1, & \text{otherwise} \end{cases}$$

is the usual Legendre symbol. Since  $f_p$  has constant coefficient 0, it is not a Littlewood polynomial, but  $g_p$  defined by  $g_p(z) := f_p(z)/z$  is a Littlewood polynomial of degree  $p - 2$ . Fekete polynomials are examined in detail in [B-02], [BE-02], [CG-00], [E-11], [E-12], [EL-07], and [M-80]. In [CE-15a] and [CE-15b] the authors examined the maximal size of the Mahler measure of sums on  $n$  monomials on the unit circle as well as on subarcs of the unit circles. In the constructions appearing in [CE-15a] properties of the Fekete polynomials  $f_p$  turned out to be quite useful. In [CG-00] B. Conrey, A. Granville, B. Poonen, and K. Soundararajan proved that for an odd prime  $p$  the Fekete polynomial  $f_p(z) = \sum_{k=0}^{p-1} \left( \frac{k}{p} \right) z^k$  (the coefficients are Legendre symbols) has  $\sim \kappa_0 p$  zeros on the unit circle, where  $0.500813 > \kappa_0 > 0.500668$ . So Fekete polynomials are far from having only  $o(p)$  zeros on the unit circle.

Mercer [M-06a] proved that if a Littlewood polynomial  $P \in \mathcal{L}_n$  of the form  $P(z) = \sum_{j=0}^n a_j z^j$  is skew-reciprocal, that is,  $a_j = (-1)^j a_{n-j}$  for each  $j = 0, 1, \dots, n$ , then it has no zeros on the unit circle. However, by using different elementary methods it was observed in both [E-01] and [M-06a] that if a Littlewood polynomial  $P$  of the form (1.1)

is self-reciprocal, that is,  $a_{j,n} = a_{n-j}$  for each  $j = 0, 1, \dots, n$ ,  $n \geq 1$ , then it has at least one zero on the unit circle.

There are many other papers on the zeros of constrained polynomials. Some of them are [BP-32], [BE-97], [BE-07], [BE-08], [BE-13], [BE-08], [B-97], [D-08], [E-08a], [E08b], [E-16a], [E-16b], [L-61], [L-64], [L-66a], [L-66b], [L-68], [M-06], [Sch-32], [Sch-33], [Sz-34], and [TV-07].

## 2. NEW RESULTS

**Theorem 2.1.** *The Rudin-Shapiro polynomials  $P_n$  and  $Q_n$  of degree  $N - 1$  with  $N := 2^n$  have  $o(N)$  zeros on the unit circle.*

The proof of Theorem 2.1 will follow by combining the recently proved Saffari's conjecture stated as Conjecture 1.1 and the theorem below. Let  $K := \mathbb{R} \pmod{2\pi}$ .

**Theorem 2.2.** *If  $p \in \mathcal{T}_n$  is of the form  $p(t) = |R(e^{it})|^2$ , where  $R \in \mathcal{P}_n^c$ , and  $p$  has at least  $k$  zeros in  $K$  (counting multiplicities), then*

$$m(\{t \in K : |p(t)| \leq \alpha \|p\|_K\}) \geq \frac{\sqrt{\alpha}}{e} \frac{k}{n}$$

for every  $\alpha \in (0, 1)$ , where  $m(A)$  denotes the one-dimensional Lebesgue measure of  $A \subset K$ .

**Theorem 2.3.** *Let  $P_n$  and  $Q_n$  be the Rudin-Shapiro polynomials of degree  $N - 1$  with  $N := 2^n$ . There is an absolute constant  $c > 0$  such that each of the functions  $\text{Re}(P_n)$ ,  $\text{Re}(Q_n)$ ,  $\text{Im}(P_n)$ , and  $\text{Im}(Q_n)$  has at least  $cN$  zeros on the unit circle.*

## 3. LEMMAS

To prove Theorem 2.1 we need the lemma below that is proved in [BE-95, E.11 of Section 5.1 on pages 236–237].

**Lemma 3.1.** *Let  $p \in \mathcal{T}_n$ ,  $t_0 \in K$ , and  $r > 0$ . Then  $p$  has at most  $enr|p(t_0)|^{-1}\|p\|_K$  zeros in the interval  $[t_0 - r, t_0 + r]$ .*

## 4. PROOFS

*Proof of Theorem 2.2.* Let  $P \in \mathcal{T}_n$  and  $Q \in \mathcal{T}_n$  be defined by

$$P(t) := \text{Re}(R(e^{it})) \quad \text{and} \quad Q(t) := \text{Im}(R(e^{it})), \quad t \in K.$$

Then

$$(4.1) \quad p(t) = |R(e^{it})|^2 = P(t)^2 + Q(t)^2, \quad t \in K.$$

Suppose  $p \in \mathcal{T}_n$  defined by  $p(t) = |R(e^{it})|^2$  has at least  $k$  zeros in  $K$ , and let  $\alpha \in (0, 1)$ . Then

$$\{t \in K : |p(t)| \leq \|p\|_K\}$$

can be written as the union of pairwise disjoint intervals  $I_j$ ,  $j = 1, 2, \dots, m$ . Each of the intervals  $I_j$  contains a point  $y_j \in I_j$  such that

$$|p(y_j)| = \alpha \|p\|_K.$$

Hence, (4.1) implies that for each  $j = 1, 2, \dots, m$ , we have either

$$(4.2) \quad |P(y_j)| \geq \left(\frac{\alpha}{2}\right)^{1/2} \|R\|_K \geq \left(\frac{\alpha}{2}\right)^{1/2} \|P\|_K$$

or

$$(4.3) \quad |Q(y_j)| \geq \left(\frac{\alpha}{2}\right)^{1/2} \|R\|_K \geq \left(\frac{\alpha}{2}\right)^{1/2} \|Q\|_K.$$

Also, each zero of  $p$  lying in  $K$  is contained in one of the intervals  $I_j$ . Let  $\mu_j$  denote the number of zeros of  $p$  lying in  $I_j$ . Since  $p \in \mathcal{T}_n$  has at least  $k$  zeros in  $K$ , so do  $P \in \mathcal{T}_n$  and  $Q \in \mathcal{T}_n$ , and we have  $\sum_{j=1}^m \mu_j \geq k$ . Note that Lemma 3.1 applied to  $P \in \mathcal{T}_n$  yields that

$$\mu_j \leq en|I_j| \left( \left(\frac{\alpha}{2}\right)^{1/2} \|P\|_K \right)^{-1} \|P\|_K = \frac{e\sqrt{2}n}{\alpha^{1/2}} |I_j|$$

for each  $j = 1, 2, \dots, m$  for which (4.2) holds. Also, Lemma 3.1 applied to  $Q \in \mathcal{T}_n$  yields that

$$\mu_j \leq en|I_j| \left( \left(\frac{\alpha}{2}\right)^{1/2} \|Q\|_K \right)^{-1} \|Q\|_K = \frac{e\sqrt{2}n}{\alpha^{1/2}} |I_j|$$

for each  $j = 1, 2, \dots, m$  for which (4.3) holds. Hence

$$\mu_j \leq \frac{e\sqrt{2}n}{\alpha^{1/2}} |I_j|, \quad j = 1, 2, \dots, m.$$

Therefore

$$k \leq \sum_{j=1}^m \mu_j \leq \frac{e\sqrt{2}n}{\alpha^{1/2}} \sum_{j=1}^m |I_j| \leq \frac{e\sqrt{2}n}{\alpha^{1/2}} m(\{t \in K : |p(t)| \leq \alpha \|p\|_K\}),$$

and the lemma follows.  $\square$

*Proof of Theorem 2.1.* We show that the Rudin-Shapiro polynomial  $P_n$  has  $o(N)$  zeros on the unit circle, where  $N = 2^n - 1$ . The proof of the fact that the Rudin-Shapiro polynomial  $Q_n$  has  $o(N)$  zeros on the unit circle is analogous. Suppose to the contrary that there are  $\varepsilon > 0$  and an increasing sequence  $(n_j)_{j=1}^\infty$  of positive integers such that the Rudin-Shapiro polynomials  $P_{n_j}$  have at least  $\varepsilon N_j$  zeros on the unit circle, where  $N_j := 2^{n_j}$  for each  $j = 1, 2, \dots$ . Then  $P_{n_j}$  has at least one zero on the unit circle and hence (1.1) and (1.2) imply that

$$(4.4) \quad \|P_{n_j}(e^{it})\|_K^2 = 2^{n_j+1}.$$

Then Theorem 2.2 implies that

$$m(\{t \in K : |P_{n_j}(t)|^2 \leq \alpha \|P_{n_j}\|_K^2\}) \geq \frac{\sqrt{\alpha}}{e} \frac{\varepsilon N_j}{N_j} = \frac{\varepsilon \sqrt{\alpha}}{e}$$

for every  $\alpha \in (0, 1)$  and  $j = 1, 2, \dots$ . Hence,

$$(4.5) \quad \limsup_{j \rightarrow \infty} m(\{t \in K : |P_{n_j}(e^{it})|^2 \leq \alpha \|P_{n_j}(e^{it})\|_K^2\}) \geq \frac{\varepsilon \sqrt{\alpha}}{e}$$

for every  $\alpha \in (0, 1)$ . On the other hand, Conjecture 1.1 proved in [B-16] combined with (4.4) imply that

$$(4.6) \quad \lim_{j \rightarrow \infty} m(\{t \in K : |P_{n_j}(e^{it})|^2 \leq \alpha \|P_{n_j}(e^{it})\|_K^2\}) = \alpha$$

for every  $\alpha \in (0, 1)$ . Combining (4.5) and (4.6) we obtain

$$\frac{\varepsilon \sqrt{\alpha}}{e} \leq \alpha,$$

that is,  $\varepsilon/e \leq \sqrt{\alpha}$  for every  $\alpha \in (0, 1)$ , a contradiction.  $\square$

*Proof of Theorem 2.3.* We prove that there is an absolute constant  $c > 0$  such that  $\text{Re}(P_n)$  has at least  $cN$  zeros on the unit circle; the fact that each of the functions  $\text{Re}(Q_n)$ ,  $\text{Im}(P_n)$ , and  $\text{Im}(Q_n)$  has at least  $cN$  zeros on the unit circle can be proved similarly. Let, as before  $K := \mathbb{R} \pmod{2\pi}$ . Let

$$\mathcal{A}_N := \left\{ f : f(t) = \sum_{j=1}^m \cos(jt + \alpha_j), \quad \alpha_j \in \mathbb{R} \right\}.$$

For an  $f \in \mathcal{A}_N$  let

$$\mu = M_2(f) := \left( \frac{1}{2\pi} \int_K |f(t)|^2 dt \right)^{1/2} = \left( \frac{N}{2} \right)^{1/2},$$

and let  $\mathcal{N}(f, v)$  be the number of real roots of  $f - v\mu$  in  $[-\pi, \pi)$ . Littlewood proves (see Theorem 1 (i) of [Li-66a]) that if  $f \in \mathcal{A}_N$  and

$$M_1(f) = \frac{1}{2\pi} \int_0^{2\pi} |f(t)| dt = c\mu,$$

then

$$\mathcal{N}(f, v) \geq 2^{-16} c^{11} N, \quad |v| \leq 2^{-5} c^3.$$

The reader may wish to find this lower bound hidden in the proof of Theorem 1 (i) of Littlewood's paper [L-66a]. Now let  $f(t) := \text{Re}(P_n)(e^{it}) - 1 \in \mathcal{A}_{N-1}$ , where  $N := 2^n$ . Observe that (1.1) implies that  $|f(t)| \leq 2N + 1$  for all  $t \in K$ , and hence

$$\begin{aligned} M_1(f) &= \frac{1}{2\pi} \int_0^{2\pi} |f(t)| dt \geq \frac{1}{2\pi} (2N + 1)^{-1/2} \int_0^{2\pi} |f(t)|^2 dt \\ &= (2N + 1)^{-1/2} \frac{N - 1}{2} = \frac{(N - 1)^{1/2}}{4} = \frac{\mu}{4}. \end{aligned}$$

Thus, by the above mentioned result of Littlewood, the trigonometric polynomial  $f(t) + 1 = \text{Re}(P_n)(e^{it})$  has at least  $2^{-16} 4^{-11} N = 2^{-38} N$  zeros in  $[-\pi, \pi)$ .  $\square$

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